2d XY Model - Kosterlitz Thouless Transition

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Preface: In these notes I discuss the 2D XY Model. Going over different limits, topological defects, and the RG analysis. Within the RG analysis I shed light on the sine-Gordon dual model, deriving the beta functions from both momentum shell RG and scaling arguments. Great references used for the development of these notes were [4, 1], with additional insight provided by [3, 2].

The Mermin-Wagner (MW) theorem ensures that continuous symmetries cannot be spontaneously broken (globally) at finite temperature in infinite systems of dimension $d \leq 2$. This is precisely the reason for why magnetism cannot exist in 1 or 2 dimensions, as magnetism consists of a certain degree of alignment of spins along a preferred direction (no continuous rotation symmetry). There does still exist the possibility of phase transitions which preserve continuous symmetries, one of which is known as the Kosterlitz-Thouless (KT) transition of the 2d XY Model

$$H = -J\sum_{\langle ij\rangle} \mathbf{S}_i \cdot \mathbf{S}_j = -J\sum_{\langle ij\rangle} \cos(\theta_i - \theta_j), \tag{1}$$

where $\mathbf{S}_i = (\cos\theta_i,\ \sin\theta_i)$ is the angle of a spin at site i relative to some ground state orientation of the system.

This model exhibits a continuous O(2) symmetry consistent with a simultaneous rotation of all spins by angle θ. The preservation of this symmetry, as a consequence of the MW-theorem, manifests itself as an absence of net magnetization $\langle S\rangle = 0$. However, spin-spin correlation functions exhibit different behavior depending on the system temperature: at high-T there is a disordered phase with an exponentially decaying correlation function, $e^{-r/\xi}$ $(r \gg \xi)$, and at low-T there is a phase of quasi-long-range order consistent with a power law decay in the correlation function, $(a/r)^{\eta}$, $(a \ll r \ll \xi)$. This is in contrast to the long-range order present in models with non-zero magnetization, where the correlation functions decay to a constant value.

The transition between these two phases is indeed marked by a divergence in the correlation length ξ, but instead of being a standard 1st or 2nd order transition, it is of infinite-order (all derivatives of the free energy are continuous across the transition). As I will show, the physical mechanism for the KT transition is that as the temperature is increased, bound (topological) vortex-antivortex pairs present at low-T unbind and collectively form a plasma of free vortices and antivortices at high-T. This is why the KT transition is often called a topological phase transition.

Low-T Phase

The continuum (low-T) limit of the model (1) is an O(2) nonlinear sigma model,

$$H[\theta] = -J\sum_{\langle ij\rangle}\left[1 - \frac{1}{2}(\theta_i - \theta_j)^2 + \cdots\right] \approx \text{const.} + \frac{J}{2}\int d^2r\ (\nabla\theta)^2, \tag{2}$$

as neighboring fluctuations are penalized. The const. term is the ground state energy of the broken-symmetry configuration where all spins are equal: $\theta_i = \theta_j$ or $\theta(r) = \text{const.}$, and so the nLSM represents soft deviations from this ground state, with $\theta(\mathbf{r})$ as the Goldstone mode. Since the theory is Gaussian (2), the correlation function between spins can be computed using Wick's identity:

$$\begin{aligned}\langle \mathbf{S}(\mathbf{r})\cdot\mathbf{S}(0)\rangle = \text{Re}\left\langle e^{i(\theta(\mathbf{r})-\theta(0))}\right\rangle &= \text{Re}\left[\frac{1}{\mathcal{Z}}\int \mathcal{D}[\theta]\exp\left(i(\theta(\mathbf{r}) - \theta(0)) - \frac{\beta J}{2}\int d^2r\ (\nabla\theta)^2\right)\right] \\ &= \text{Re}\left[\frac{1}{\mathcal{Z}}\int \mathcal{D}[\theta]\exp\left(\int \frac{d^2k}{(2\pi)^2}\left[i\theta(\mathbf{k})(e^{i\mathbf{k}\cdot\mathbf{r}} - 1) - \frac{\beta J}{2}k^2\theta(\mathbf{k})\theta(-\mathbf{k})\right]\right)\right].\end{aligned} \tag{3}$$

Completing the square on the inside,

$$
\begin{aligned}
-\frac{\beta J k^2}{2}\left(\theta(\mathbf{k})\theta(-\mathbf{k}) - \frac{2i}{\beta J k^2}\theta(\mathbf{k})(e^{i\mathbf{k}\cdot\mathbf{r}}-1)\right) &= -\frac{\beta J k^2}{2}\Big(\theta(\mathbf{k})\theta(-\mathbf{k}) - \frac{i}{\beta J k^2}\theta(\mathbf{k})(e^{i\mathbf{k}\cdot\mathbf{r}}-1) \\
&\qquad - \frac{i}{\beta J k^2}\theta(-\mathbf{k})(e^{-i\mathbf{k}\cdot\mathbf{r}}-1)\Big) \\
&= -\frac{\beta J k^2}{2}\left(\theta(\mathbf{k}) - \frac{i}{\beta J k^2}(e^{-i\mathbf{k}\cdot\mathbf{r}}-1)\right)\left(\theta(-\mathbf{k}) - \frac{i}{\beta J k^2}(e^{i\mathbf{k}\cdot\mathbf{r}}-1)\right) \\
&\qquad - \frac{1}{2\beta J k^2}(e^{-i\mathbf{k}\cdot\mathbf{r}}-1)(e^{i\mathbf{k}\cdot\mathbf{r}}-1).
\end{aligned} \tag{4}
$$

where the first equality is valid under the integral sign, results in

$$
\begin{aligned}
\langle \mathbf{S}(\mathbf{r})\cdot\mathbf{S}(0)\rangle &= \exp\left(-\frac{1}{2\beta J}\int \frac{d^2k}{(2\pi)^2}\frac{1}{k^2}(e^{-i\mathbf{k}\cdot\mathbf{r}}-1)(e^{i\mathbf{k}\cdot\mathbf{r}}-1)\right) \\
&\times \mathrm{Re}\left[\frac{1}{\mathcal{Z}}\int \mathcal{D}[\theta]\exp\left(-\frac{\beta J}{2}\int\frac{d^2k}{(2\pi)^2}k^2\left(\theta(\mathbf{k}) - \frac{i}{\beta J k^2}(e^{-i\mathbf{k}\cdot\mathbf{r}}-1)\right)\left(\theta(-\mathbf{k}) - \frac{i}{\beta J k^2}(e^{i\mathbf{k}\cdot\mathbf{r}}-1)\right)\right)\right].
\end{aligned} \tag{5}
$$

Redefining the angular variable to compensate the shifts, the remaining term is simply $\mathcal{Z}$, which cancels with that already present in the denominator:

$$
\begin{aligned}
\langle \mathbf{S}(\mathbf{r})\cdot\mathbf{S}(0)\rangle &= \exp\left(-\frac{1}{2\beta J}\int \frac{d^2k}{(2\pi)^2}\frac{1}{k^2}(e^{-i\mathbf{k}\cdot\mathbf{r}}-1)(e^{i\mathbf{k}\cdot\mathbf{r}}-1)\right) \\
&\quad\times \mathrm{Re}\left[\frac{1}{\mathcal{Z}}\int \mathcal{D}[\theta]\exp\left(-\frac{\beta J}{2}\int\frac{d^2k}{(2\pi)^2}k^2\theta(\mathbf{k})\theta(-\mathbf{k})\right)\right] \\
&= \exp\left(-\frac{1}{2\beta J}\int \frac{d^2k}{(2\pi)^2}\frac{1}{k^2}(e^{-i\mathbf{k}\cdot\mathbf{r}}-1)(e^{i\mathbf{k}\cdot\mathbf{r}}-1)\right) \\
&= \mathrm{Re}\left[\exp\left(-\frac{1}{\beta J}\int\frac{d^2k}{(2\pi)^2}\frac{1-e^{i\mathbf{k}\cdot\mathbf{r}}}{k^2}\right)\right].
\end{aligned} \tag{6}
$$

I want to investigate this in the long distance limit $r\to\infty$. For small k the integral converges, but at large k the integral diverges. A large k cutoff can be introduced by a lattice regularization a, and for large $\mathbf{r}$ I find

$$
\langle \mathbf{S}(\mathbf{r})\cdot\mathbf{S}(0)\rangle = \exp\left(-\frac{1}{2\pi\beta J}\int_0^{1/a}dk\;k\int_0^{2\pi}\frac{d\theta}{2\pi}\frac{1-e^{ikr\cos\theta}}{k^2}\right) = \exp\left(-\frac{1}{2\pi\beta J}\int_0^{1/a}dk\;\frac{1-J_0(kr)}{k}\right). \tag{7}
$$

Evaluating the integral carefully,

$$
\begin{aligned}
\int_0^{1/a}dk\;\frac{1-J_0(kr)}{k} = \int_0^{r/a}du\;\frac{1-J_0(u)}{u} &= \int_0^1 du\;\frac{1-J_0(u)}{u} - \int_1^{r/a}du\;\frac{J_0(u)}{u} + \int_1^{r/a}\frac{du}{u} \\
&= -0.116 + \ln\left(\frac{r}{a}\right),
\end{aligned} \tag{8}
$$

where I let $1/a\to\infty$ in the second integral only. Ignoring the constant term in the large r limit, I find

$$
\langle \mathbf{S}(\mathbf{r})\cdot\mathbf{S}(0)\rangle = \left(\frac{a}{r}\right)^{\eta}, \qquad \eta = \frac{1}{2\pi\beta J} \qquad \text{(Low-T)} \tag{9}
$$

or quasi-long range order. A similar calculation in $d=1$ gives exponential decay in the correlator $e^{-r/a}$, whereas in $d=3$ the correlator decays to a constant value. In the latter case, there will be long-range order and a certain degree of alignment of spins (ordered phase) at low temperature. In fact, for $d\geq 3$ the model will exhibit the usual ferro-para magnetic transition.

High-T Phase

In this case, $\beta J \ll 1$ and the partition function can be expanded in $\kappa = \beta J$,

$$\mathcal{Z} = \int \mathcal{D}[\theta] \exp\left(\kappa \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j)\right) = \prod_i \int_0^{2\pi} \frac{d\theta_i}{2\pi} \prod_{\langle ij \rangle} \left[1 + \kappa \cos(\theta_i - \theta_j) + \mathcal{O}(\kappa^2)\right]. \tag{10}$$

I again want to determine the spin-spin correlator. Each term in the product can be represented by a "bond" that connects sites i and j, each contributing a factor of 1 or $\kappa \cos(\theta_i - \theta_j)$. Since

$$\int_0^{2\pi} d\theta_i \cos(\theta_i - \theta_j) = 0 \quad \forall j \neq i, \tag{11}$$

any graph with a single bond emanating from a site vanishes. This means the only contributing terms to the partition function (10) are loops. It is important to note that

$$\int_0^{2\pi} \frac{d\theta_2}{2\pi} \cos(\theta_1 - \theta_2) \cos(\theta_2 - \theta_3) = \frac{1}{2} \cos(\theta_1 - \theta_3), \tag{12}$$

and so each bond integration gives 1/2 times the cosine of the difference between the end points. This means that to compute $\cos(\theta_r - \theta_0)$, I need to only take terms connecting θ_0 with θ_r:

$$\begin{aligned}
\langle \mathbf{S}(\mathbf{r}) \cdot \mathbf{S}(0) \rangle = \langle \cos(\theta_r - \theta_0) \rangle &= \int_0^{2\pi} \frac{d\theta_0 \cdots d\theta_r}{(2\pi)^r} \kappa \cos(\theta_0 - \theta_1) \kappa \cos(\theta_1 - \theta_2) \cdots \kappa \cos(\theta_{r-1} - \theta_r) \cos(\theta_r - \theta_0) \\
&= \frac{\kappa^r}{2^{r-1}} \int_0^{2\pi} \frac{d\theta_0 d\theta_r}{(2\pi)^2} \cos^2(\theta_r - \theta_0) \\
&= \left(\frac{\kappa}{2}\right)^r,
\end{aligned} \tag{13}$$

or simply

$$\langle \mathbf{S}(\mathbf{r}) \cdot \mathbf{S}(0) \rangle = e^{-r/\xi}, \qquad \xi^{-1} = \ln(2/\kappa) \qquad \text{(High-T)} \tag{14}$$

This exponential decay in the high-T phase is indicative of a disordered phase. Somehow the quasi-long-range order of the low-T phase is disordered with an increase in temperature without the breaking of O(2) symmetry. The mechanism for this disorder happens to be something I neglected in the above calculation, the fact that θ is defined mod 2π.

Vortices

So, I've excluded the fact that θ is defined mod 2π, and by treating θ in this single-valued manner I have derived a power-law correlation between spins at low-T. Take a field configuration $\{\theta_j\}$ where the phase winds an integer n of a full 2π rotation over some closed path j around an anchor point; these are vortices of topological charge n. $n = +1$ is a vortex, while $n = -1$ is an anti-vortex Fig.[1]. The discreteness of n makes a continuous deformation returning the vortex to a non-winding configuration impossible. It's easy to see that θ_j varies strongly near the center of the vortex, i.e. that the center of the vortex is not accessible to continuum approximations with a globally continuous field θ.

Far from the center, θ becomes a smooth variable and continuum methods can be performed: the integral

$$\oint d\mathbf{l} \cdot \nabla\theta = 2\pi n \tag{15}$$

along a path enclosing the vortex yields its charge. (15) seemingly violates Stokes' Theorem, but note that θ is multi-valued. This is better elucidated by defining $\mathbf{u} \equiv \nabla\theta$, such that

$$\begin{aligned}
2\pi n &= \oint d\mathbf{l} \cdot \mathbf{u} \\
&= \iint d\mathbf{a} \cdot (\nabla \times \mathbf{u}) \iff \nabla \times \mathbf{u} = 2\pi \sum_i n_i \delta^2(\mathbf{r} - \mathbf{r}_i),
\end{aligned} \tag{16}$$

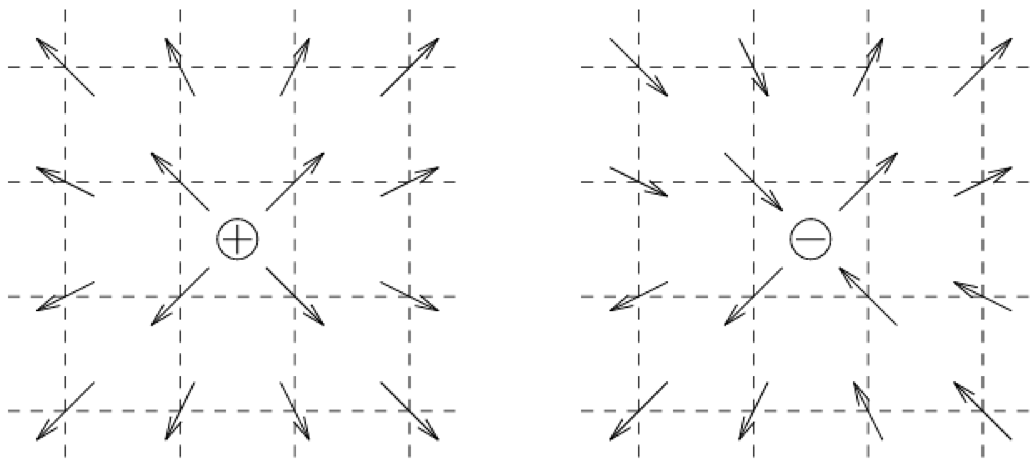

Figure 1: Vortices of topological charge ± 1, showing the winding of the phase around the vortex center.

where in the last step I allowed for multiple charges n_i at positions $\mathbf{r}_i$. I will further re-express $\mathbf{u}$ into single-valued functions via a Helmholtz decomposition,

$$\mathbf{u} = \nabla\phi - \nabla\times(\psi\hat{z}). \tag{17}$$

Here, ϕ is the "spin-wave" part with power law correlation previously investigated, and the second term is the "vorticity" with ψ as the vortex field obeying from (16) the condition

$$\nabla^2\psi = 2\pi\sum_i n_i\delta^2(\mathbf{r}-\mathbf{r}_i) \iff \psi = \sum_i n_i \ln|\mathbf{r}-\mathbf{r}_i| \tag{18}$$

since $\nabla^2\psi = \nabla\times(\nabla\times(\psi\hat{z}))$. Note from (18) that the solution to ψ is the same as the $2d$ Coulomb potential, which is a comparison I will expand on later. I can now re-express the low-T nLSM Hamiltonian (2),

$$H = \frac{J}{2}\int d^2r\,(\nabla\theta)^2 = \frac{J}{2}\int d^2r\,\left[(\nabla\phi)^2 - 2\nabla\phi\cdot\nabla\times(\psi\hat{z}) + (\nabla\times(\psi\hat{z}))^2\right], \tag{19}$$

where the second term vanishes by an IBP, and the first term is the usual spin-wave term giving rise to the $r^{-\kappa}$ correlation. The new "vortex" term is simply

$$\begin{aligned} H_{\text{vortex}} = \frac{J}{2}\int d^2r\,(\nabla\times(\psi\hat{z}))^2 = -\frac{J}{2}\int d^2r\,\psi\nabla^2\psi &= -\pi J\int d^2r\,\sum_i n_i\psi\delta^2(\mathbf{r}-\mathbf{r}_i) \\ &= -\pi J\sum_i n_i\psi(\mathbf{r}_i) \\ &= -\pi J\sum_{i,j} n_i n_j \ln|\mathbf{r}_i-\mathbf{r}_j| \end{aligned} \tag{20}$$

where in the last step I used (16). Hence, vortices/anti-vortices with topological charges n_i interact in the same way electric charges interact in 2d. Note that at the precise location of a vortex, θ is not well-defined. I will admit ignorance on short-distance scales and say the vortex has some "core" size a which will be used as a UV cutoff.

A basic configuration with winding number n is $\nabla\theta = (n/r)\hat{\phi}$, where $\hat{\phi}$ is the azimuthal unit vector. The (single) vortex action can then be written

$$\begin{aligned} S_n = S_n^{\text{core}} + \frac{\kappa}{2}\int_a d^2r\,(\nabla\theta)^2 &= S_n^{\text{core}} + \pi\kappa n^2\int_a^L \frac{dr}{r} \\ &= S_n^{\text{core}} + \pi\kappa n^2 \ln\left(\frac{L}{a}\right), \end{aligned} \tag{21}$$

where S_n^{core} are details inside core. The log-term dominates the action and diverges with system size, inhibiting spontaneous formation of vortices at low-T which preserves the quasi-long-range order.

Focusing on more energetically favored configurations with, say $n = 1$, the number of ways of placing a single vortex of size a in a system of size L is simply $(L/a)^2$, since we are in 2d. The partition function for a single vortex may then be written

$$\mathcal{Z}_1 = \left(\frac{L}{a}\right)^2 \exp\left(-S_1^{\text{core}} - \pi\kappa \ln\left(\frac{L}{a}\right)\right) = \exp\left(-S_1^{\text{core}} - (\pi\kappa - 2)\ln\left(\frac{L}{a}\right)\right). \tag{22}$$

Hence, the entropy change with a single vortex is $2\ln(L/a)$, and the energy of a single vortex is $\pi J \ln(L/a)$. The phase transition occurs when these two equate, giving

$$T_c = \pi J/2 \tag{23}$$

as the critical temperature for the KT-transition.

For $T < T_c$, energy outweighs entropy and no free vortices are present: they are bound together as vortex/anti-vortex pairs whose number diminishes with decreasing temperature. Once $T > T_c$, entropy outweighs energy cost for creating vortices, and they become statistically probable. Vortices then unbind and proliferate the system, forming a 2d plasma of topological charges which interact in the same manner as electric charges in 2d.

Renormalization Group Analysis

Sine-Gordon Model

The above description is quite rough, and is best refined through a more rigorous RG analysis. This naturally accounts for the screening of the (bare) logarithmic interaction between vortices due to the presence of mobile charge carriers. By nature of the 2d Coulomb gas analogy, it is illuminating to first reformulate the action as a sine-Gordon model, another model in the BKT universality class. I will work in the lattice description to begin; take a lattice with spacing a and lattice sites $\mathbf{X}_\alpha$, where each site possesses a variable $n_\alpha = \{-1, 0, 1\}$ representing an anti-vortex, an empty site, and a vortex.[1] The choice in these three values ensures that vortices have a large repulsion (hard core) and that vortices and anti-vortices annihilate when they come too close. Hence, the partition function may be written

$$\mathcal{Z}_{\text{vortex}} = \sum_{\{n_\alpha\}} \exp\left(-\sum_\alpha n_\alpha^2 S_{\text{core}} + \pi\kappa \sum_{\alpha\neq\beta} n_\alpha n_\beta \ln\left(\frac{|\mathbf{X}_\alpha - \mathbf{X}_\beta|}{a}\right)\right), \tag{24}$$

where I am restricting the sum $\sum_{\{n_\alpha\}}$ to neutral configurations $\sum_\alpha n_\alpha = 0$. Now, since the Green's function of the 2d Laplacian is $(1/2\pi)\ln|\mathbf{r}|$, I can write

$$\exp\left(-\frac{1}{2}\int d^2x d^2y\, f(\mathbf{x})\frac{1}{2\pi}\ln|\mathbf{x}-\mathbf{y}|\, f(\mathbf{y})\right) \sim \int \mathcal{D}[\phi] \exp\left(-\int d^2x\, \left(\frac{1}{2}(\nabla\phi)^2 + f(\mathbf{x})\phi(\mathbf{x})\right)\right), \tag{25}$$

or in our case,

$$\exp\left(\pi\kappa \sum_{\alpha\neq\beta} n_\alpha \ln\left(\frac{|\mathbf{X}_\alpha - \mathbf{X}_\beta|}{a}\right) n_\beta\right) \sim \int \mathcal{D}[\phi] \exp\left(-\int d^2x\, \frac{1}{2}(\nabla\phi)^2 + 2\pi i\sqrt{\kappa}\sum_\alpha n_\alpha \phi(\mathbf{X}_\alpha)\right), \tag{26}$$

[1] α is the site number, and doesn't parameterize the vortex type $(-1, 0, 1)$.

where I left the term independent of n_α as continuous.[2] The reason for this is apparent when substituting this result into (24),

$$
\begin{aligned}
\mathcal{Z}_{\text{vortex}} &= \int \mathcal{D}[\phi] \sum_{\{n_\alpha\}} \exp\left(-\int d^2x\, \frac{1}{2}(\nabla\phi)^2 + \sum_\alpha (-n_\alpha^2 S_{\text{core}} + 2\pi i\sqrt{\kappa} n_\alpha \phi(\mathbf{X}_\alpha))\right) \\
&= \int \mathcal{D}[\phi] \exp\left(-\int d^2x\, \frac{1}{2}(\nabla\phi)^2\right) \prod_\alpha \sum_{n_\alpha=-1,0,1} \exp\left(-n_\alpha^2 S_{\text{core}} + 2\pi i\sqrt{\kappa} n_\alpha \phi(\mathbf{X}_\alpha)\right) \\
&= \int \mathcal{D}[\phi] \exp\left(-\int d^2x\, \frac{1}{2}(\nabla\phi)^2\right) \prod_\alpha \left(1 + 2e^{-S_{\text{core}}} \cos\left(2\pi\sqrt{\kappa}\phi(\mathbf{X}_\alpha)\right)\right) \\
&\approx \int \mathcal{D}[\phi] \exp\left(-\int d^2x\, \frac{1}{2}(\nabla\phi)^2 + \frac{2e^{-S_{\text{core}}}}{a^2} a^2 \sum_\alpha \cos\left(2\pi\sqrt{\kappa}\phi(\mathbf{X}_\alpha)\right)\right) \\
&\approx \int \mathcal{D}[\phi] \exp\left(-\int d^2x\, \left(\frac{1}{2}(\nabla\phi)^2 - \frac{2e^{-S_{\text{core}}}}{a^2} \cos\left(2\pi\sqrt{\kappa}\phi\right)\right)\right),
\end{aligned}
\tag{27}
$$

i.e., the sum over configurations is simpler in the lattice representation. Note that in the 4th equality I assumed the fugacities were small to re-express the cosine term as an exponential.

Defining the parameters $\lambda_0 \equiv 2e^{-S_{\text{core}}}/a^2$ and $\beta_0 \equiv 2\pi\sqrt{\kappa}$, I find the famous sine-Gordon model

$$
\mathcal{Z}_{\text{vortex}} = \int \mathcal{D}[\phi] \exp\left(-\int d^2x\, \left(\frac{1}{2}(\nabla\phi)^2 - \lambda_0 \cos(\beta_0\phi)\right)\right),
\tag{28}
$$

where the parameter λ_0, containing the core energies S_{core}, is related to the vortex fugacity, while β_0 is related to the coupling between vortices.

Momentum Shell RG

I will first break the action into

$$
S_0[\phi] = \int d^2x\, \frac{1}{2}(\nabla\phi)^2, \qquad\qquad S_1[\phi] = -\int d^2x\, \lambda_0 \cos(\beta_0\phi)
\tag{29}
$$

where I will treat S_1 perturbatively. I will furthermore break up the field into fast and slow components, $\phi_>$ and $\phi_<$, where $\phi_>$ is defined on momentum scales $\Lambda' < k < \Lambda$, with $\Lambda = 1/a$ being the original UV cutoff. The goal is to integrate out the fast field $\phi_>$, generating an effective action for the slow mode.

Since S_0 is diagonal in momentum space, the effective action follows as

$$
\begin{aligned}
\mathcal{Z}_\Lambda = \int \mathcal{D}\phi_< \mathcal{D}\phi_>\, e^{-S_0[\phi_<]-S_0[\phi_>]-S_1[\phi_<+\phi_>]} &= \int \mathcal{D}\phi_>\, e^{-S_0[\phi_>]} \int \mathcal{D}\phi_<\, e^{-S_0[\phi_<]} \frac{\mathcal{D}\phi_>\, e^{-S_0[\phi_>]-S_1[\phi_<+\phi_>]}}{\int \mathcal{D}\phi_>\, e^{-S_0[\phi_>]}} \\
&= \mathcal{Z}_> \int \mathcal{D}\phi_<\, e^{-S_0[\phi_<]} \left\langle e^{-S_1[\phi_<+\phi_>]} \right\rangle_> \\
&= \mathcal{Z}_> \int \mathcal{D}\phi_<\, e^{-S_{\text{eff}}[\phi_<]},
\end{aligned}
\tag{30}
$$

where (valid under small fugacities λ_0),

$$
\begin{aligned}
S_{\text{eff}}[\phi_<] = S_0[\phi_<] - \ln\left\langle e^{-S_1[\phi_<+\phi_>]} \right\rangle_> &\approx S_0[\phi_<] - \ln\left(1 - \langle S_1 \rangle_> + \frac{1}{2}\langle S_1^2 \rangle_>\right) \\
&\approx S_0[\phi_<] + \langle S_1 \rangle_> - \frac{1}{2}\left(\langle S_1^2 \rangle_> - \langle S_1 \rangle_>^2\right)
\end{aligned}
\tag{31}
$$

where I noted that the log subtracts the disconnected diagrams (or used the cumulant expansion) in writing the additional term. All that is left is to compute the averages.

[2] Note here $\int d^2x \sim a^2 \sum_\alpha$

Computing the First Order Average (31)

Here, we have that

$$\begin{aligned}\langle S_1[\phi_< + \phi_>]\rangle_> = -\lambda_0 \int d^2x \ \langle \cos[\beta_0(\phi_< + \phi_>)]\rangle_> &= -\lambda_0 \ \mathrm{Re} \int d^2x \ \langle \exp\left[i\beta_0(\phi_< + \phi_>)\right]\rangle_> \\ &= -\lambda_0 \ \mathrm{Re} \int d^2x \ \exp\left(-\frac{\beta_0^2}{2} \langle \phi_>(x)\phi_>(x)\rangle_>\right) \exp\left(i\beta_0\phi_<\right),\end{aligned} \tag{32}$$

where I used Wick's identity since S_0 is Gaussian. The remaining average is just the propagator for the first mode evaluated at the same point,

$$\langle \phi_>(x)\phi_>(x)\rangle_> = \int_{\Lambda'}^{\Lambda} \frac{d^2k}{(2\pi)^2}\frac{1}{k} = \frac{1}{2\pi}\ln\left(\frac{\Lambda}{\Lambda'}\right), \tag{33}$$

and so (34) may be written

$$\begin{aligned}\langle S_1[\phi_< + \phi_>]\rangle_> &= -\lambda_0 \ \mathrm{Re} \int d^2x \ \exp\left(-\frac{\beta_0^2}{4\pi}\ln\left(\frac{\Lambda}{\Lambda'}\right)\right) \exp\left(i\beta_0\phi_<\right) \\ &= -\lambda_0 \left(\frac{\Lambda}{\Lambda'}\right)^{-\beta_0^2/4\pi} \int d^2x \ \cos\left(\beta_0\phi_<\right) \\ &= -\lambda_0 \xi^{-\beta_0^2/4\pi} \int d^2x \ \cos\left(\beta_0\phi_<\right),\end{aligned} \tag{34}$$

where I defined $\Lambda' = \Lambda/\xi$. I now need to restore the cutoff, $x' = x/\xi$,

$$\langle S_1[\phi_< + \phi_>]\rangle_> = -\lambda_0 \xi^{2-\beta_0^2/4\pi} \int d^2x \ \cos\left(\beta_0\phi_<(\xi x)\right). \tag{35}$$

Ignoring for now the rescaling of the field ϕ (which is trivial at this order) it's clear to see that this term renormalizes λ,

$$\lambda(\xi) = \lambda_0 \xi^{2-\beta_0^2/4\pi}. \tag{36}$$

In the case that $\beta_0^2 > 8\pi$, λ is irrelevant (vanishes with length scale) and the action has just gradient terms with the power law correlation discussed previously. If $\beta_0^2 < 8\pi$, λ becomes relevant, growing with length scale. In such a case, the minimum of the potential is $\phi = 0 \bmod 2\pi/\beta_0$, and expanding the potential around that point gives a quadratic contribution in ϕ, leading to exponentially decaying correlations (as seen above in the high-T limit).

Note that the critical value $\beta_0^2 = 8\pi$ can be rewritten as $T_c = \pi J/2$, which is precisely the same critical temperature (23) found above. This relation will not be quite true to second order in λ. Note also that since $\kappa = 2/\pi$ at the transition, the low-T (power-law) scaling dimension (9) is $\eta = 1/4$.

Computing the Second Order Averages (31)

Here, we have that

$$
\begin{aligned}
\frac{1}{2}\left(\left\langle S_1^2\right\rangle_> - \left\langle S_1\right\rangle_>^2\right) = \frac{\lambda_0^2}{2}\int d^2x d^2y \Big[&\langle \cos(\beta_0(\phi_<(x)+\phi_>(x)))\cos(\beta_0(\phi_<(y)+\phi_>(y)))\rangle_> \\
&- \langle \cos(\beta_0(\phi_<(x)+\phi_>(x)))\rangle_> \langle \cos(\beta_0(\phi_<(y)+\phi_<(y)))\rangle_> \Big] \\
= \frac{\lambda_0^2}{4}\int d^2x d^2y \Big\{ &\cos(\beta_0(\phi_<(x)+\phi_<(y)))\Big[\exp\left(-\frac{\beta_0^2}{2}\left\langle(\phi_>(x)+\phi_>(y))^2\right\rangle\right) \\
&- \exp\left(-\frac{\beta_0^2}{2}\left\langle\phi_>^2(x)\right\rangle\right)\exp\left(-\frac{\beta_0^2}{2}\left\langle\phi_>^2(y)\right\rangle\right)\Big] \\
&+ \cos(\beta_0(\phi_<(x)-\phi_<(y)))\Big[\exp\left(-\frac{\beta_0^2}{2}\left\langle(\phi_>(x)-\phi_>(y))^2\right\rangle\right) \\
&- \exp\left(-\frac{\beta_0^2}{2}\left\langle\phi_>^2(x)\right\rangle\right)\exp\left(-\frac{\beta_0^2}{2}\left\langle\phi_>^2(y)\right\rangle\right)\Big]\Big\}
\end{aligned} \tag{37}
$$

which is derived by expanding in terms of exponentials, separating fast/slow modes, and using Wick's identity. Each of the $\left\langle \phi_>^2\right\rangle$ terms contribute a $\xi^{-\beta_0^2/4\pi}$ outside the integral, and the propagator at different points can be written $\langle\phi_>(x)\phi_>(y)\rangle = G(x-y)$, giving

$$
\begin{aligned}
\frac{1}{2}\left(\left\langle S_1^2\right\rangle_> - \left\langle S_1\right\rangle_>^2\right) = \frac{\lambda_0^2}{4}\xi^{-\beta_0^2/2\pi}\int d^2x d^2y \Big\{ &\cos(\beta_0(\phi_<(x)+\phi_<(y)))\left[e^{-\beta_0^2 G(x-y)}-1\right] \\
&+ \cos(\beta_0(\phi_<(x)-\phi_<(y)))\left[e^{\beta_0^2 G(x-y)}-1\right]\Big\}.
\end{aligned} \tag{38}
$$

Note from the expression of the propagator

$$
G(x-y) = \int_{\Lambda'}^{\Lambda} \frac{d^2k}{(2\pi)^2}\frac{e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}}{k}, \tag{39}
$$

that oscillations of the integrand are fast for $|\mathbf{x}-\mathbf{y}| \gg 1/\Lambda'$, in which case G averages to zero. Hence, I am forced to small $|\mathbf{x}-\mathbf{y}|$, justifying the expansion under $\mathbf{y} = \mathbf{x}+\mathbf{v}$ for small $\mathbf{v}$. Thus,

$$
\begin{aligned}
&\cos(\beta_0(\phi_<(x)+\phi_<(x+v))) \approx \cos(2\beta_0\phi_<(x)), \\
&\cos(\beta_0(\phi_<(x)-\phi_<(x+v))) \approx \cos(\beta_0(-\mathbf{v}\cdot\nabla\phi_<(x))) \approx 1 - \frac{\beta_0^2 v^2}{2}(\nabla\phi_<(x))^2,
\end{aligned} \tag{40}
$$

such that

$$
\begin{aligned}
\frac{1}{2}\left(\left\langle S_1^2\right\rangle_> - \left\langle S_1\right\rangle_>^2\right) = \frac{\lambda_0^2}{4}\xi^{-\beta_0^2/2\pi}\int d^2x d^2v \Big[&\cos(2\beta_0\phi_<(x))\left(e^{-\beta_0^2 G(v)}-1\right) \\
&+ \left(e^{\beta_0^2 G(v)}-1\right) - \left(e^{\beta_0^2 G(v)}-1\right)\frac{\beta_0^2 v^2}{2}(\nabla\phi_<(x))^2\Big] \\
= \text{const.} + \frac{\lambda_0^2}{2}\int d^2x \left[A_1(\xi)\cos(2\beta_0\phi_<(x)) + A_2(\xi)(\nabla\phi_<(x))^2\right],&
\end{aligned} \tag{41}
$$

where I've defined the cutoff dependent constants

$$
A_1(\xi) = \frac{1}{2}\xi^{-\beta_0^2/2\pi}\int d^2v\ (e^{-\beta_0^2 G(v)}-1), \qquad A_2(\xi) = -\frac{1}{4}\beta_0^2\xi^{-\beta_0^2/2\pi}\int d^2v\ v^2(e^{\beta_0^2 G(v)}-1). \tag{42}
$$

The A_1 term is a two vortex interaction, and the A_2 term renormalizes the stiffness β. The two vortex interaction was not included in the original model, and won't be needed here, as I care only about the evolution of the parameters.

The next step is to restore the cutoff, $x' = x/\xi$, and I find

$$\frac{1}{2}\left(\langle S_1^2\rangle_> - \langle S_1\rangle_>^2\right) = \text{const.} + \frac{\lambda_0^2}{2}A_2(\xi)\int d^2x\,(\nabla\phi_<(\xi x))^2, \tag{43}$$

such that the effective action (31) may be written

$$S_{\text{eff}} = \int d^2x\left[\frac{1}{2}(\nabla\phi_<(\xi x))^2 - \lambda_0\xi^{2-\beta_0^2/4\pi}\cos(\beta_0\phi_<(\xi x)) + \frac{\lambda_0^2}{2}A_2(\xi)(\nabla\phi_<(\xi x))^2\right] \tag{44}$$

to second order in λ^2, neglecting constants and two vortex interactions. In order for the kinetic term to remain canonically normalized, the field ϕ must be rescaled. The correct choice is

$$\begin{aligned}(1+\lambda_0^2A_2(\xi))(\nabla\phi_<(\xi x))^2 \to (\nabla\phi'_<(x))^2 \iff \phi_<(\xi x) &\to (1+\lambda_0^2A_2(\xi))^{-1/2}\phi'_<(x)\\ &\approx (1-\frac{\lambda_0^2}{2}A_2(\xi))\phi'_<(x),\end{aligned} \tag{45}$$

which has the effect of renormalizing β,

$$S_{\text{eff}} = \int d^2x\left[\frac{1}{2}(\nabla\phi_<(x))^2 - \lambda_0\xi^{2-\beta_0^2/4\pi}\cos(\beta_0(1-\frac{\lambda_0^2}{2}A_2(\xi))\phi_<(x))\right]. \tag{46}$$

Beta Functions and RG Flow

At long last, the renormalized parameters are written from (46) as

$$\lambda(\xi) = \lambda_0\xi^{2-\beta_0^2/4\pi}, \qquad\qquad \beta(\xi) = \beta_0\left(1-\frac{\lambda_0^2}{2}A_2(\xi)\right), \tag{47}$$

where $A_2(\xi)$ is defined in (42). In deriving the beta functions, I write $\xi = e^s$ for small s. This immediately gives

$$\frac{d\lambda}{ds} = \left(2-\frac{\beta^2}{4\pi}\right)\lambda, \tag{48}$$

for the fugacity parameter. Note that since $A_2(\xi) \propto \xi^{-\beta_0^2/2\pi}$,

$$\frac{d\beta}{ds} = \frac{\lambda^2\beta^3}{4\pi}A_2(\beta) \equiv -C(\beta)\beta^3\lambda^2, \tag{49}$$

where $C(\beta)$ is some positive coefficient, and its exact value is unimportant.[3] This is because λ can be rescaled freely in (49) to absorb any pre-factors without effect in (48).

I will first rewrite the beta function (49) for $\kappa^{-1} = 4\pi^2/\beta^2$, in which case I find

$$\frac{d\lambda}{ds} = (2-\pi\kappa)\,\lambda, \qquad\qquad \frac{d\kappa^{-1}}{ds} = 8\pi^2C\lambda^2. \tag{50}$$

Last, I will zoom into the phase transition at $\kappa = 2/\pi$ by defining the new variables $x = \kappa^{-1} - \pi/2$ and $y = \sqrt{2\pi C}\lambda$ ($x, y \ll 1$):

$$\frac{dx}{ds} = 4y^2, \qquad\qquad \frac{dy}{ds} = \frac{4}{\pi}xy. \tag{51}$$

The above form for the KT RG flow equations is that which is most commonly recognized. In xy space, the flows lie along hyperbolae, as (51) implies

$$x^2 - \pi y^2 = c, \tag{52}$$

where c is some constant. The flows are sketched by hand in Fig.[2].

[3] The choice in isolating β^3 is purely for convenience.

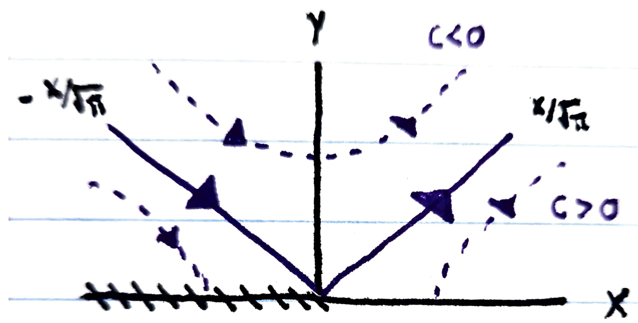

Figure 2: RG flows for parameters x, y with relation (52).

For $c < 0$, the foci are on the y-axis, and for $c > 0$ the foci are on the x-axis. In the low-T (or low x) case, for $c > 0$ a small y_0 terminates at a fixed line $y = 0$, $\kappa^{-1} \leq \pi/2$. Larger y_0 needn't necessarily terminate, and will asymptote to smaller κ and larger y where PT eventually breaks down; this would be the high-T vortex proliferation phase. In fact, for $\kappa^{-1} > \pi/2$, or $x > 0$, any value of c leads to vortex proliferation.

It is natural to think that $c \propto T_c - T$, where T_c is the critical temperature. At the transition we have $c = 0$, which translates to

$$x_c^2 = \pi y_c^2 \iff \kappa_c^{-1} - \pi/2 = -\sqrt{\pi} y_c \iff T_c = \frac{\pi J}{2} - \sqrt{\pi} J y_0, \tag{53}$$

and so a small, non-zero y_0 reduces the critical temperature predicted above by $-\sqrt{\pi} J y_0$. In the case that $c > 0$ for $x_0 < 0$, in the thermodynamic limit $s \to \infty$ we have $y \to 0$ and $x \to -b\sqrt{T_c - T}$, where b is some positive constant. This allows me write an effective stiffness in the vicinity of the transition,

$$\kappa_{\text{eff}} = \left(\frac{\pi}{2} - b\sqrt{T_c - T}\right)^{-1} \approx \frac{2}{\pi} + \frac{4b}{\pi}\sqrt{T_c - T}. \tag{54}$$

Experiments on superfluid thin films confirm a $2/\pi$ jump in κ at the critical point, as well as square root scaling **??**.

Lastly, the correlation length ξ can be determined by integrating the beta functions (51) explicitly. For $c < 0$ flows that don't quite hit the $y = 0$ axis, I can use $y^2 = \pi^{-2}(x^2 + |c|)$ to write

$$\frac{dx}{ds} = \frac{4}{\pi^2}(|c| + x^2) \iff s = \frac{\pi}{4\sqrt{|c|}} \tan^{-1}\left(\frac{x(s)}{\sqrt{|c|}}\right) - \frac{\pi}{4\sqrt{|c|}} \tan^{-1}\left(\frac{x(0)}{\sqrt{|c|}}\right). \tag{55}$$

The subtracted term is just a constant, and can be ignored by choosing $x(0) = 0$. By the time $x(s) \sim 1$, then $y(s) \sim 1$ as well and the theory is gapped and I can stop the RG flow here. As the transition is approached from above, $|c| = T - T_c \ll 1$, and I can approximate $\tan^{-1}(1/\sqrt{|c|}) \sim \pi/2$. From (55), this gives

$$s = \frac{\pi^2}{8\sqrt{|c|}} + \text{const.} \implies \xi = e^s \sim \exp\left(\frac{1}{\sqrt{T - T_c}}\right), \tag{56}$$

and the correlation length is strongly divergent for $T = T_c$, stronger than the soft (power-law) divergences in correlation length usually seen in phase transitions. Since the free energy is dimensionless and extensive, it must scale like $F \sim (L/\xi)^2$, and so

$$F \sim \exp\left(-\frac{1}{\sqrt{T - T_c}}\right), \tag{57}$$

which is a very weak singularity, as there is no discontinuity in <u>any</u> derivative of the free energy: the KT transition is a phase transition of infinite order.

Scaling Approach to the Sine-Gordon Model

The 2d XY model action at low-T is

$$S[\theta] = \int d^2x \, \frac{\beta J}{2} (\nabla \theta)^2. \tag{58}$$

The variable θ is defined mod 2π, and hence can possess topological defects (vortices) with winding 2π. To capture this, in the dual sine-Gordon field theory we introduce a term $g\cos(2\pi\tilde{\theta})$, where g is the vortex fugacity and $\tilde{\theta}$ is the dual field. The temperature absorbed stiffness βJ is inverted in the dual theory, and hence

$$S_{\text{dual}}[\tilde{\theta}] = \int d^2x \, \left[\frac{1}{2\beta J} (\nabla \tilde{\theta})^2 - g\cos(2\pi\tilde{\theta}) \right], \tag{59}$$

is the dual sine-Gordon action.[4] The phase boundaries are easy to determine by scaling arguments [2]. First we rescale $\tilde{\theta} \to \sqrt{\beta J}\tilde{\theta}$ so that $[\tilde{\theta}] = 0$. This augments the cosine term as $g\cos(2\pi\sqrt{\beta J}\tilde{\theta})$, and its scaling dimension should be -2 to cancel the integration measure. We can determine the scaling dimension Δ of the cosine operator,

$$\left\langle \exp\left(2\pi i \sqrt{\beta J}\tilde{\theta}_0\right) \exp\left(-2\pi i \sqrt{\beta J}\tilde{\theta}_r\right) \right\rangle = \left(\frac{1}{|r/a|^{\pi\beta J}} \right)^2 \iff \Delta = \pi\beta J, \tag{60}$$

where I used Wick's identity (since Gaussian) then recognized the Green's function of the 2d Laplacian (which is logarithmic). This corresponds to a β-function of the form

$$\frac{dg}{ds} = (2 - \pi\beta J) g, \tag{61}$$

as above, with the phase transition occurring when g is marginal: $2 = \pi J/T$. When g is relevant, $\tilde{\theta}$ is pinned to integer values (59), and hence the fluctuations are massive (gapped). Gapped fluctuations screen the vortex/anti-vortex interaction at long-distances, freeing up vortex/anti-vortex pairs.

Summary

The classical XY model has the form

$$H = -J \sum_{\langle ij \rangle} \mathbf{S}_i \cdot \mathbf{S}_j = -J \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j), \tag{62}$$

where $\mathbf{S}_i = (\cos\theta_i, \sin\theta_i)$ is the angle of a spin at site i. The continuum (low-T) limit is an $O(2)$ nonlinear sigma model, and the spin-spin correlation function exhibits power law decay with distance (quasi long-range order). For high-T, one performs a loop expansion and finds exponential decay of the spin-spin correlation function indicative of a disordered phase. The transition between the two behaviors is marked by a exponential divergence in the correlation length, rather than a power law, and is known as the Kosterlitz-Thouless transition. This is a phase transition of ∞-order, so all derivatives of the free energy are continuous across the transition. Physically, as T increases, bound topological vortices confined at low-T will unbind at high-T to form a Coulomb gas (plasma) of free vortices/antivortices with logarithmic interactions.

The phase boundaries are determined from the RG behavior of the model, and since the 2d XY model is dual to a sine-Gordon field theory, the beta functions are

$$\frac{d(\beta J)^{-1}}{ds} = 4\pi^3 y^2, \qquad \frac{dy}{ds} = (2 - \pi\beta J) y, \tag{63}$$

where y is related to the fugacity of vortices. In the vicinity of the critical point $T_c = \pi J/2$, the RG flows in the $y - J$ plane are hyperbolae with a fixed line $y = 0$, $T < \pi J/2$ corresponding to bound vortex/antivortex pairs.

[4] This is derived by performing a Villain transformation on (62) (F.T. of bonds), integrating out θ to generate an action in terms of an integer-valued field, then using the Poisson summation formula to generate the spin-wave portion $\tilde{\theta}$ and the vortices $m(r)$. Summing over vortex configurations gives (59). See [3] for details.

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